

N 64 32993
 (ACCESSION NUMBER)
 OR-58998
 (NASA CR OR TMX OR AD NUMBER)
 (THRU)
 (CODE)
 28
 (CATEGORY)

Technical Report No. 32-322

*Use of the Frobenius Series in Solving Homogeneous
 Linear Systems of Differential Equations
 With Weak Singular Points*

Russell E. Carr

OTS PRICE

XEROX \$ 1.00 FS
 MICROFILM \$ 0.50 mf

jpl

JET PROPULSION LABORATORY
 CALIFORNIA INSTITUTE OF TECHNOLOGY
 PASADENA, CALIFORNIA

August 27, 1962

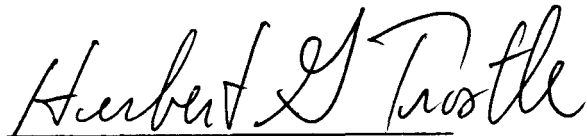
02-08-5694

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CONTRACT No. NAS 7-100

Technical Report No. 32-322

***Use of the Frobenius Series in Solving Homogeneous
Linear Systems of Differential Equations
With Weak Singular Points***

Russell E. Carr

A handwritten signature in cursive script, reading "Herbert G. Trostle". The signature is written in dark ink and is positioned above a horizontal line.

Herbert G. Trostle, Acting Chief
Research Analysis Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

August 27, 1962

Copyright© 1962
Jet Propulsion Laboratory
California Institute of Technology

ABSTRACT

32993

Although the use of Frobenius Series in finding solutions of a homogeneous linear differential equation with weak singular points is well known, the extension of this method to systems of such equations is not ordinarily discussed in texts written in English.

This Report is written primarily for expository purposes. The method is first illustrated with an elementary example, then applied to a specific problem which arose in connection with another study.

The importance of obtaining the complete solution is discussed.



CONTENTS

I. Introduction	1
II. An Elementary Example	2
III. Application to a System of Three Second-Order Equations	4
IV. Concluding Remarks	7

I. INTRODUCTION

The homogeneous linear differential equation

$$p_0(x) \frac{d^n y}{dx^n} + p_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1}(x) \frac{dy}{dx} + p_n(x) y = 0 \quad (1)$$

is said to have a weak singular point at $x = 0$ providing the ratios

$$\frac{p_m(x)}{p_0(x)} \quad (m = 1, 2, \dots, n) \quad (2)$$

are representable in the neighborhood of $x = 0$ as

$$\frac{p_m(x)}{p_0(x)} = \frac{1}{x^m} \sum_{k=0}^{\infty} a_{mk} x^k \quad (3)$$

where all the series

$$\sum_{k=0}^{\infty} a_{mk} x^k \quad (4)$$

converge in the neighborhood of $x = 0$.

A well known approach to the problem of finding a solution to Eq. 1 in the neighborhood of $x = 0$ is to try the Frobenius Series

$$y = \sum_{j=0}^{\infty} c_j x^{j+r} \quad c_0 \neq 0 \quad (5)$$

The use of Frobenius Series in the case of a homogeneous linear system of differential equations with a weak singular point at $x = 0$ does not appear to be as well known.

Consider the homogeneous linear system

$$x \frac{dy_p}{dx} = \sum_{q=1}^n f_{p,q}(x) y_q \quad p = 1, 2, \dots, n \quad (6)$$

where the $f_{p,q}(x)$ are representable in the neighborhood of $x = 0$ by

$$f_{p,q}(x) = \sum_{k=0}^{\infty} a_{p,q}^{(k)} x^k \quad (7)$$

and where not all the $a_{p,q}^{(0)}$ have the value zero. The solution of this problem is discussed by Kamke.¹

The generalization of Eq. 5 is to assume

$$y_p = x^r \sum_{j=0}^{\infty} C_{p,j} x^j \quad p = 1, 2, \dots, n \quad (8)$$

where not all the $C_{p,0}$ vanish at the same time. The use of this method is illustrated by the elementary example of Section II.

A standard approach to solving a homogeneous linear system of differential equations of higher order is to transform this system to one of differential equations of the first order and then solve the latter system. However, in fact, it may be simpler to approach the system of higher order equations directly.

Section III treats a specific problem which the author encountered in connection with another study. This problem involves solution of a homogeneous linear system of three second-order differential equations with a weak singular point at the origin. The author found the system of second-order equations simpler to work with than the corresponding linear system of first-order equations.

This Report has been written primarily for expository purposes. It was prompted largely by the fact that the subject matter is not ordinarily discussed in texts written in English.

¹Kamke, E., *Differentialgleichungen Lösungsmethoden und Lösungen*, Band I, "Gewöhnliche Differentialgleichungen," 3 Auflage, Chelsea Publishing Company, New York, 1948.

II. AN ELEMENTARY EXAMPLE

Consider the system of equations

$$\begin{aligned} dy/dx &= (a/x)y + (b/x)z \\ dz/dx &= (d/x)z \end{aligned} \quad (9)$$

where

$$abd \neq 0, \quad a \neq d \quad (10)$$

The general solution to Eqs. 9 can be found by using the formula for first-order, first-degree, linear differential equations. This solution is given by

$$\begin{aligned} y &= C_1 x^a + \frac{bC_2}{d-a} x^d \\ z &= C_2 x^d \end{aligned} \quad (11)$$

where C_1 and C_2 are arbitrary constants.

To arrive at the general solution using the Frobenius series approach, assume

$$\begin{aligned} y &= x^r \sum_{k=0}^{\infty} A_k x^k \\ z &= x^r \sum_{k=0}^{\infty} B_k x^k \end{aligned} \quad (12)$$

where not both A_0 and B_0 vanish.

Substituting from Eqs. 12 into Eqs. 9 gives

$$\begin{aligned} \sum_{k=0}^{\infty} [(r+k-a)A_k - bB_k] x^{r+k} &= 0 \\ \sum_{k=0}^{\infty} [(r+k-d)B_k] x^{r+k} &= 0 \end{aligned} \quad (13)$$

Hence

$$\begin{aligned} (r-a)A_0 - bB_0 &= 0 \\ (r-d)B_0 &= 0 \end{aligned} \quad (14)$$

and, since not both A_0 and B_0 can vanish,

$$\begin{vmatrix} (r-a) & -b \\ 0 & (r-d) \end{vmatrix} = 0 \quad (15)$$

Because of the assumptions of Eqs. 10, Eq. 15 has two distinct roots,

$$r_1 = a, \quad r_2 = d \quad (16)$$

First consider the root $r_1 = a$. Equations 13 become

$$\begin{aligned} \sum_{k=0}^{\infty} [kA_k^{(1)} - bB_k^{(1)}] x^{a+k} &= 0 \\ \sum_{k=0}^{\infty} [(k+a-d)B_k^{(1)}] x^{a+k} &= 0 \end{aligned} \quad (13a)$$

Hence

$$\begin{aligned} -bB_0^{(1)} &= 0 \\ (a-d)B_0^{(1)} &= 0 \end{aligned} \quad (17)$$

and

$$\begin{aligned} B_0^{(1)} &= 0 \\ A_0^{(1)} &\neq 0 \end{aligned} \quad (18)$$

Further, for $k = 1, 2, \dots$

$$\begin{aligned} kA_k^{(1)} - bB_k^{(1)} &= 0 \\ (k+a-d)B_k^{(1)} &= 0 \end{aligned} \quad (19)$$

Thus, if $d-a$ is not a positive integer,

$$A_k^{(1)} = B_k^{(1)} = 0 \quad (20)$$

while, if $d-a$ is a positive integer M , $B_M^{(1)}$ is arbitrary and

$$\begin{aligned} A_M^{(1)} &= (bB_M^{(1)}/M) \\ A_k^{(1)} = B_k^{(1)} &= 0 \quad \text{for } k \neq M \end{aligned} \quad (21)$$

Thus, the root $r_1 = a$ gives the following solution

$$\begin{aligned} y_1 &= A_0^{(1)} x^a \\ z_1 &= 0 \end{aligned} \quad (22)$$

if $d-a$ is not a positive integer, and gives the solution

$$\begin{aligned} y_1 &= A_0^{(1)} x^a + (bB_M^{(1)}/M) x^{a+M} \\ z_1 &= B_M^{(1)} x^{a+M} \end{aligned} \quad (23)$$

if $d-a$ is a positive integer M .

Next, consider the root $r_2 = d$. Equations 13 become

$$\begin{aligned} \sum_{k=0}^{\infty} [(k+d-a)A_k^{(2)} - bB_k^{(2)}] x^{d+k} &= 0 \\ \sum_{k=0}^{\infty} [kB_k^{(2)}] x^{d+k} &= 0 \end{aligned} \quad (13b)$$

Hence,

$$(d-a)A_0^{(2)} - bB_0^{(2)} = 0 \quad (24)$$

and

$$B_0^{(2)} \neq 0, \quad A_0^{(2)} = bB_0^{(2)}/(d-a) \quad (25)$$

Further, for $k = 1, 2, \dots$

$$\begin{aligned} (k+d-a)A_k^{(2)} - bB_k^{(2)} &= 0 \\ kB_k^{(2)} &= 0 \end{aligned} \quad (26)$$

III. APPLICATION TO A SYSTEM OF THREE SECOND-ORDER EQUATIONS

In connection with another study, the author encountered a homogeneous linear system of six first-order differential equations with a weak singular point at the origin. This system had been derived from a homogeneous linear system of three second-order equations. Although the derived equations were simpler to handle numerically when not in the neighborhood of the origin, the necessity for investigating the behavior of the solutions in that neighborhood suggested looking at Frobenius series solutions. Working with the original system proved simpler than treating the derived system.

The original system can be considered as a special case of the following:

$$\begin{aligned} \frac{d^2 y_i}{dx^2} = & \frac{a_i}{x} \frac{dy_1}{dx} + \frac{b_i}{x} \frac{dy_2}{dx} + \frac{c_i + d_i x^2}{x} \frac{dy_3}{dx} \\ & + \frac{e_i + f_i x^2}{x^2} y_1 + \frac{g_i + h_i x^2}{x^2} y_2 + \frac{i_i + j_i x^2}{x^2} y_3 \end{aligned} \quad (37)$$

where $i = 1, 2, 3$.

Assuming

$$\begin{aligned} y_1 &= x^r \sum_{k=0}^{\infty} A_k x^k \\ y_2 &= x^r \sum_{k=0}^{\infty} B_k x^k \\ y_3 &= x^r \sum_{k=0}^{\infty} C_k x^k \end{aligned} \quad (38)$$

where A_0 , B_0 , and C_0 do not all vanish, substitution of expressions of Eqs. 38 into Eqs. 37 gives, if one defines

$$A_{-2} = A_{-1} = B_{-2} = B_{-1} = C_{-2} = C_{-1} = 0 \quad (39)$$

the following three equations, for $k = 0, 1, 2, \dots$

$$\begin{aligned} & \{[(r+k)(r+k-1) - a_1(r+k) - e_1] A_k - f_1 A_{k-2}\} \\ & + \{[-b_1(r+k) - g_1] B_k - h_1 B_{k-2}\} \\ & + \{[-c_1(r+k) - i_1] C_k \\ & + [-d_1(r+k-2) - j_1] C_{k-2}\} = 0 \end{aligned} \quad (40)$$

$$\begin{aligned} & \{[-a_2(r+k) - e_2] A_k - f_2 A_{k-2}\} \\ & + \{[(r+k)(r+k-1) - b_2(r+k) - g_2] B_k - h_2 B_{k-2}\} \\ & + \{[-c_2(r+k) - i_2] C_k \\ & + [-d_2(r+k-2) - j_2] C_{k-2}\} = 0 \end{aligned} \quad (41)$$

$$\begin{aligned} & \{[-a_3(r+k) - e_3] A_k - f_3 A_{k-2}\} \\ & + \{[-b_3(r+k) - g_3] B_k - h_3 B_{k-2}\} \\ & + \{[(r+k)(r+k-1) - c_3(r+k) - i_3] C_k \\ & + [-d_3(r+k-2) - j_3] C_{k-2}\} = 0 \end{aligned} \quad (42)$$

In particular, for $k = 0$, the requirement that A_0 , B_0 , and C_0 do not all vanish leads to the result that r must be a solution of the equation

$$\begin{vmatrix} r(r-1) - a_1 r - e_1 & -b_1 r - g_1 & -c_1 r - i_1 \\ -a_2 r - e_2 & r(r-1) - b_2 r - g_2 & -c_2 r - i_2 \\ -a_3 r - e_3 & -b_3 r - g_3 & r(r-1) - c_3 r - i_3 \end{vmatrix} = 0 \quad (43)$$

For the specific problem encountered,

$$\begin{aligned} a_1 &= -2 & a_2 &= -\frac{\lambda + \mu}{\mu} & a_3 &= 4\pi G\rho \\ b_1 &= \left(\frac{\lambda + \mu}{\lambda + 2\mu}\right) n(n+1) & b_2 &= -2 & b_3 &= 0 \\ c_1 &= 0 & c_2 &= 0 & c_3 &= -4 \\ d_1 &= -\frac{\rho a^2}{\lambda + 2\mu} & d_2 &= 0 & d_3 &= 0 \end{aligned}$$

Thus, if $a - d$ is not a positive integer,

$$\begin{aligned} B_k^{(2)} &= 0 \\ A_k^{(2)} &= 0 \end{aligned} \quad (27)$$

while, if $a - d$ is a positive integer N ,

$$\begin{aligned} B_k^{(2)} &= 0 \\ A_k^{(2)} &= 0 \quad \text{for } k \neq N \end{aligned} \quad (28)$$

and $A_N^{(2)}$ is arbitrary.

Thus, the root $r_2 = d$ gives the solution

$$\begin{aligned} y_2 &= [bB_0^{(2)}/(d-a)] x^d \\ z_2 &= B_0^{(2)} x^d \end{aligned} \quad (29)$$

when $a - d$ is not a positive integer, and gives the solution

$$\begin{aligned} y_2 &= A_N^{(2)} x^{d+N} - (bB_0^{(2)}/N) x^d \\ z_2 &= B_0^{(2)} x^d \end{aligned} \quad (30)$$

if $a - d$ is a positive integer N .

Since the system of Eqs. 9 is linear, its general solution is given by

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ z &= c_1 z_1 + c_2 z_2 \end{aligned} \quad (31)$$

where c_1 and c_2 are arbitrary constants.

Thus, if $|d - a|$ is not an integer, the solution to the system of Eqs. 9 is given by

$$\begin{aligned} y &= c_1 A_0^{(1)} x^a + c_2 [bB_0^{(2)}/(d-a)] x^d \\ z &= c_2 B_0^{(2)} x^d \end{aligned} \quad (32)$$

which, since $A_0^{(1)} \neq 0$, $B_0^{(2)} \neq 0$, and c_1 and c_2 are arbitrary constants, agrees with the general solution given by Eqs. 11.

If $d - a$ is a positive integer M , the solution to the system of Eqs. 9 is given by

$$\begin{aligned} y &= c_1 [A_0^{(1)} x^a + (bB_M^{(1)}/M) x^{a+M}] + c_2 (bB_0^{(2)}/M) x^{a+M} \\ z &= c_1 B_M^{(1)} x^{a+M} + c_2 B_0^{(2)} x^{a+M} \end{aligned} \quad (33)$$

or

$$\begin{aligned} y &= (c_1 A_0^{(1)}) x^a + (b/M) (c_1 B_M^{(1)} + c_2 B_0^{(2)}) x^{a+M} \\ z &= (c_1 B_M^{(1)} + c_2 B_0^{(2)}) x^{a+M} \end{aligned} \quad (34)$$

which, since $A_0^{(1)} \neq 0$, $B_0^{(2)} \neq 0$, $B_M^{(1)}$ is arbitrary, c_1 and c_2 are arbitrary, and $M = d - a$, also agrees with the general solution (Eqs. 11).

If $a - d$ is a positive integer N , the solution to the system of Eqs. 9 is given by

$$\begin{aligned} y &= c_1 A_0^{(1)} x^a + c_2 [A_N^{(2)} x^{d+N} - (bB_0^{(2)}/N) x^d] \\ z &= c_2 B_0^{(2)} x^d \end{aligned} \quad (35)$$

or

$$\begin{aligned} y &= (c_1 A_0^{(1)} + c_2 A_N^{(2)}) x^a - (c_2 b B_0^{(2)}/N) x^d \\ z &= c_2 B_0^{(2)} x^d \end{aligned} \quad (36)$$

which, since $A_0^{(1)} \neq 0$, $B_0^{(2)} \neq 0$, $A_N^{(2)}$ is arbitrary, c_1 and c_2 arbitrary, $N = a - d$, also agrees with the general solution (Eqs. 11).

Since all possibilities lead to solutions which can be written in the form of the general solution as expressed by Eqs. 11, the Frobenius Series approach has been shown to lead to the general solution of the system of Eqs. 9.

$$\begin{aligned}
e_1 &= 2 + \frac{\mu}{\lambda + 2\mu} n(n+1) & e_2 &= -\frac{2(\lambda + 2\mu)}{\mu} & e_3 &= 8\pi G\rho \\
f_1 &= -\frac{\sigma^2 \rho a^2 + (4/3)\pi G\rho^2 a^2}{\lambda + 2\mu} & f_2 &= \frac{(4/3)\pi G\rho^2 a^2}{\mu} & f_3 &= 0 \\
g_1 &= -\left(\frac{\lambda + 3\mu}{\lambda + 2\mu}\right) n(n+1) & g_2 &= \frac{\lambda + 2\mu}{\mu} n(n+1) & g_3 &= -4\pi G\rho n(n+1) \\
h_1 &= \frac{(4/3)\pi G\rho^2 a^2 n(n+1)}{\lambda + 2\mu} & h_2 &= -\frac{\sigma^2 \rho a^2}{\mu} & h_3 &= 0 \\
i_1 &= 0 & i_2 &= 0 & i_3 &= n^2 + n - 2 \\
j_1 &= -\frac{\rho a^2}{\lambda + 2\mu} & j_2 &= -\frac{\rho a^2}{\mu} & j_3 &= 0
\end{aligned}$$

where $\lambda, \mu, \rho, \sigma, a, G$ are constants and n is a positive integer. The equation corresponding to Eq. 43 can be factored into

$$(r - n + 1)^2 (r + n + 2)^2 (r - n - 1) (r + n) = 0 \quad (44)$$

giving the roots

$$r_1 = r_2 = n - 1, \quad r_3 = r_4 = -n - 2, \quad r_5 = n + 1, \quad r_6 = -n \quad (45)$$

The original problem included among its boundary conditions the requirement that the variable y_1, y_2 , and y_3 be regular at the origin. The roots r_3, r_4 , and r_6 are thus eliminated from further consideration.

The roots r_1 and r_2 give for the coefficients the relations

$$A_0^{(1)} = n B_0^{(1)} \quad (46)$$

$C_0^{(1)}$ and $B_0^{(1)}$ arbitrary except that both do not vanish,

$$A_1^{(1)} = B_1^{(1)} = C_1^{(1)} = 0 \quad (47)$$

$B_2^{(1)}$ is arbitrary, (48)

$$\begin{aligned}
A_2^{(1)} &= \frac{[(n^2 + n)\lambda + (n^2 - n - 2)\mu] B_2^{(1)}}{[(n + 3)\lambda + (n + 5)\mu]} \\
&+ \frac{[(4/3)\pi G\rho^2 a^2 n - \sigma^2 \rho a^2] B_0^{(1)} - \rho a^2 C_0^{(1)}}{[(n + 3)\lambda + (n + 5)\mu]} \quad (49)
\end{aligned}$$

$$C_2^{(1)} = \frac{4\pi G\rho [(n + 3) A_2^{(1)} - (n^2 + n) B_2^{(1)}]}{2(2n + 3)} \quad (50)$$

and, for $k = 3, 4, \dots$, $A_k^{(1)}$ and $B_k^{(1)}$ are determined from the simultaneous equations

$$\begin{aligned}
&\{[(n + k + 1)(n + k - 2)]\lambda \\
&+ [2(n + k)(n + k - 1) - n(n + 1) - 4]\mu\} A_k^{(1)} \\
&- \{[(n + k - 2)\lambda + (n + k - 4)\mu] n(n + 1)\} B_k^{(1)} \quad (51) \\
&= -[(4/3)\pi G\rho^2 a^2 + \sigma^2 \rho a^2] A_{k-2}^{(1)} \\
&+ (4/3)\pi G\rho^2 a^2 n(n + 1) B_{k-2}^{(1)} - \rho a^2 (n + k - 2) C_{k-2}^{(1)}
\end{aligned}$$

$$\begin{aligned}
&\{(n + k + 1)\lambda + (n + k + 3)\mu\} A_k^{(1)} \\
&- \{[n(n + 1)]\lambda + [n(n + 1) - (k - 1)(2n + k)]\mu\} B_k^{(1)} \quad (52) \\
&= (4/3)\pi G\rho^2 a^2 A_{k-2}^{(1)} - \rho \sigma^2 a^2 B_{k-2}^{(1)} - \rho a^2 C_{k-2}^{(1)}
\end{aligned}$$

while $C_k^{(1)}$ is given by

$$C_k^{(1)} = \frac{4\pi G\rho [(n + k + 1) A_k^{(1)} - (n^2 + n) B_k^{(1)}]}{k(2n + k + 1)} \quad (53)$$

The determinant of the coefficient matrix on the left side of Eqs. 51 and 52 is

$$k(2n + k + 1)(2n + k - 1)(k - 2)\mu(\lambda + 2\mu) \quad (54)$$

Therefore, the system can be solved for $A_k^{(1)}$ and $B_k^{(1)}$ as long as $k > 2$.

It is seen that, for odd k , all $A_k^{(1)}, B_k^{(1)}$, and $C_k^{(1)}$ vanish while, for even k , all $A_k^{(1)}, B_k^{(1)}, C_k^{(1)}$ can be expressed as linear functions of $B_0^{(1)}, C_0^{(1)}$, and $B_2^{(1)}$.

One solution of the original system of differential equations is then given, assuming convergence, by

$$\begin{aligned}
y_1^{(1)} &= x^{n-1} \sum_{k=0}^{\infty} A_k^{(1)} x^k \\
y_2^{(1)} &= x^{n-1} \sum_{k=0}^{\infty} B_k^{(1)} x^k \quad (55) \\
y_3^{(1)} &= x^{n-1} \sum_{k=0}^{\infty} C_k^{(1)} x^k
\end{aligned}$$

The root $r_5 = n + 1$ gives the following:

$$B_0^{(5)} \neq 0 \quad (56)$$

$$A_0^{(5)} = \frac{[(n^2 + n)\lambda + (n^2 - n - 2)\mu] B_0^{(5)}}{[(n + 3)\lambda + (n + 5)\mu]} \quad (57)$$

$$C_0^{(5)} = \frac{4\pi G\rho [(n + 3)A_0^{(5)} - (n^2 + n)B_0^{(5)}]}{2(2n + 3)} \quad (58)$$

Defining

$$A_{-2}^{(5)} = B_{-2}^{(5)} = C_{-2}^{(5)} = A_{-1}^{(5)} = B_{-1}^{(5)} = C_{-1}^{(5)} = 0 \quad (59)$$

then the solutions for r_5 can be written

$$\begin{aligned} y_1^{(5)} &= x^{n+1} \sum_{k=-2}^{\infty} A_k^{(5)} x^k \\ y_2^{(5)} &= x^{n+1} \sum_{k=-2}^{\infty} B_k^{(5)} x^k \\ y_3^{(5)} &= x^{n+1} \sum_{k=-2}^{\infty} C_k^{(5)} x^k \end{aligned} \quad (60)$$

By renaming the coefficients $A_k^{(5)}$, $B_k^{(5)}$, $C_k^{(5)}$ as $\mathcal{A}_{k+2}^{(5)}$, $\mathcal{B}_{k+2}^{(5)}$, $\mathcal{C}_{k+2}^{(5)}$, respectively, the solution for r_5 can be written

$$\begin{aligned} y_1^{(5)} &= x^{n-1} \sum_{k=0}^{\infty} \mathcal{A}_k^{(5)} x^k \\ y_2^{(5)} &= x^{n-1} \sum_{k=0}^{\infty} \mathcal{B}_k^{(5)} x^k \\ y_3^{(5)} &= x^{n-1} \sum_{k=0}^{\infty} \mathcal{C}_k^{(5)} x^k \end{aligned} \quad (61)$$

where

$$\mathcal{A}_0^{(5)} = n \mathcal{B}_0^{(5)} \quad (62)$$

$\mathcal{C}_0^{(5)}$ and $\mathcal{B}_0^{(5)}$ are both zero

$$\mathcal{A}_1^{(5)} = \mathcal{B}_1^{(5)} = \mathcal{C}_1^{(5)} = 0$$

$$\mathcal{B}_2^{(5)} \neq 0 \quad (64)$$

$$\begin{aligned} \mathcal{A}_2^{(5)} &= \frac{[(n^2 + n)\lambda + (n^2 - n - 2)\mu] \mathcal{B}_2^{(5)}}{[(n + 3)\lambda + (n + 5)\mu]} \\ &+ \frac{[(4/3)\pi G\rho^2 a^2 n - \sigma^2 \rho a^2] \mathcal{B}_0^{(5)} - \rho a^2 \mathcal{C}_0^{(5)}}{[(n + 3)\lambda + (n + 5)\mu]} \end{aligned} \quad (65)$$

$$\mathcal{C}_2^{(5)} = \frac{4\pi G\rho [(n + 3) \mathcal{A}_2^{(5)} - (n^2 + n) \mathcal{B}_2^{(5)}]}{2(2n + 3)} \quad (66)$$

and the recursion formulas relating $\mathcal{A}_k^{(5)}$, $\mathcal{B}_k^{(5)}$, $\mathcal{C}_k^{(5)}$ to $\mathcal{A}_{k-2}^{(5)}$, $\mathcal{B}_{k-2}^{(5)}$, $\mathcal{C}_{k-2}^{(5)}$ ($k = 3, 4, \dots$) are identical to those

relating $A_k^{(1)}$, $B_k^{(1)}$, $C_k^{(1)}$ to $A_{k-2}^{(1)}$, $B_{k-2}^{(1)}$, $C_{k-2}^{(1)}$. It is seen, then, that for odd k , all $\mathcal{A}_k^{(5)}$, $\mathcal{B}_k^{(5)}$, $\mathcal{C}_k^{(5)}$ vanish, while for even k , all $\mathcal{A}_k^{(5)}$, $\mathcal{B}_k^{(5)}$, $\mathcal{C}_k^{(5)}$ are expressible as linear functions of $\mathcal{B}_0^{(5)}$, $\mathcal{C}_0^{(5)}$, and $\mathcal{B}_2^{(5)}$. It is important to note that these linear functions are identical with those by which $A_k^{(1)}$, $B_k^{(1)}$, $C_k^{(1)}$ are expressible in terms of $B_0^{(1)}$, $C_0^{(1)}$, and $B_2^{(1)}$.

Since the system of differential equations is linear, it is also satisfied by

$$y_i = c_1 y_i^{(1)} + c_5 y_i^{(5)} \quad i = 1, 2, 3 \quad (67)$$

where c_1 and c_5 are arbitrary constants.

Thus,

$$\begin{aligned} y_1 &= x^{n-1} \sum_{k=0}^{\infty} (c_1 A_k^{(1)} + c_5 \mathcal{A}_k^{(5)}) \\ y_2 &= x^{n-1} \sum_{k=0}^{\infty} (c_1 B_k^{(1)} + c_5 \mathcal{B}_k^{(5)}) \\ y_3 &= x^{n-1} \sum_{k=0}^{\infty} (c_1 C_k^{(1)} + c_5 \mathcal{C}_k^{(5)}) \end{aligned} \quad (68)$$

where

$$(c_1 A_0^{(1)} + c_5 \mathcal{A}_0^{(5)}) = n(c_1 B_0^{(1)} + c_5 \mathcal{B}_0^{(5)}) \quad (69)$$

and

$$(c_1 B_0^{(1)} + c_5 \mathcal{B}_0^{(5)}) \equiv c_1 B_0^{(1)} \quad (70)$$

$$(c_1 C_0^{(1)} + c_5 \mathcal{C}_0^{(5)}) \equiv c_1 C_0^{(1)} \quad (71)$$

are arbitrary,

$$\begin{aligned} (c_1 A_1^{(1)} + c_5 \mathcal{A}_1^{(5)}) &= (c_1 B_1^{(1)} + c_5 \mathcal{B}_1^{(5)}) \\ &= (c_1 C_1^{(1)} + c_5 \mathcal{C}_1^{(5)}) = 0 \end{aligned} \quad (72)$$

$$(c_1 B_2^{(1)} + c_5 \mathcal{B}_2^{(5)}) \quad (73)$$

is arbitrary. Further, defining

$$\begin{aligned} \mathcal{A}_k &= c_1 A_k^{(1)} + c_5 \mathcal{A}_k^{(5)} \\ \mathcal{B}_k &= c_1 B_k^{(1)} + c_5 \mathcal{B}_k^{(5)} \\ \mathcal{C}_k &= c_1 C_k^{(1)} + c_5 \mathcal{C}_k^{(5)} \end{aligned} \quad (74)$$

it is seen that the system of differential equations is satisfied by

$$\begin{aligned} y_1 &= x^{n-1} \sum_{k=0}^{\infty} \mathcal{A}_k x^k \\ y_2 &= x^{n-1} \sum_{k=0}^{\infty} \mathcal{B}_k x^k \\ y_3 &= x^{n-1} \sum_{k=0}^{\infty} \mathcal{C}_k x^k \end{aligned} \quad (75)$$

where

$$\mathcal{B}_0, \mathcal{C}_0, \text{ and } \mathcal{B}_2 \quad (76)$$

are arbitrary,

$$\mathcal{A}_0 = n\mathcal{B}_0 \quad (77)$$

$$\mathcal{A}_1 = \mathcal{B}_1 = \mathcal{C}_1 = 0 \quad (78)$$

$$\begin{aligned} \mathcal{A}_2 = & \frac{[(n^2 + n)\lambda + (n^2 - n - 2)\mu] \mathcal{B}_2}{[(n + 3)\lambda + (n + 5)\mu]} \\ & + \frac{[(4/3)\pi G\rho^2 a^2 n - \sigma^2 \rho a^2] \mathcal{B}_0 - \rho a^2 \mathcal{C}_0}{[(n + 3)\lambda + (n + 5)\mu]} \end{aligned} \quad (79)$$

$$\mathcal{C}_2 = \frac{4\pi G\rho [(n + 3)\mathcal{A}_2 - (n^2 + n)\mathcal{B}_2]}{2(2n + 3)} \quad (80)$$

and, for $k = 3, 4, \dots$, the recursion formulas for \mathcal{A}_k and \mathcal{B}_k in terms of \mathcal{A}_{k-2} , \mathcal{B}_{k-2} , \mathcal{C}_{k-2} are the same as those given in Eqs. 51 and 52 relating $A_k^{(1)}$, $B_k^{(1)}$ to $A_{k-2}^{(1)}$, $B_{k-2}^{(1)}$, $C_{k-2}^{(1)}$ while the formula for \mathcal{C}_k , paralleling Eq. 53, is simply

$$\mathcal{C}_k = \frac{4\pi G\rho [(n + k + 1)\mathcal{A}_k - (n^2 + n)\mathcal{B}_k]}{k(2n + k + 1)} \quad (81)$$

Thus, for all odd k , $\mathcal{A}_k = \mathcal{B}_k = \mathcal{C}_k = 0$, while for all even k , \mathcal{A}_k , \mathcal{B}_k , and \mathcal{C}_k are expressible as linear functions of \mathcal{B}_0 , \mathcal{C}_0 , and \mathcal{B}_2 , which are now arbitrary independent constants.

It can be shown that the roots r_3 , r_4 , and r_6 bring in an additional set of three arbitrary independent constants which must be set equal to zero in order to satisfy the requirement of regularity at the origin. Thus, the solution described above is the complete solution to the problem.

IV. CONCLUDING REMARKS

The equations discussed in Section III arose in connection with the study of the free oscillations of a gravitating solid sphere. Should the Moon have a solid core and should the free modes of oscillation of the Moon be excited, then, if recorded by seismic instruments, comparison of the recorded modes with the calculated modes would provide information regarding the internal structure of the Moon. However, omission of some of the theoretically calculated modes could make it difficult to obtain a valid comparison between recorded and calcu-

lated results. Therefore, it is important that a complete solution of the differential equations be obtained.

To obtain a complete solution with the Frobenius Series method, it is necessary that the solution, before imposition of boundary conditions, contain as many arbitrary independent constants as the order of the linear system.

This requirement is fulfilled in both the elementary example and in the application in Section III.